

On the spherical derivative of a rational function

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Abstract

For a rational functions f we consider the norm of the derivative with respect to the spherical metric and denote by $K(f)$ the supremum of this norm. We give estimates of this quantity $K(f)$ both for an individual function and for sequences of iterates.

Keywords: rational function, spherical derivative, characteristic exponent.

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A rational function is a holomorphic map from the Riemann sphere into itself. We equip the Riemann sphere with the usual spherical metric whose length and area elements are

$$ds = \frac{|dz|}{1 + |z|^2} \quad \text{and} \quad dA = \frac{dxdy}{(1 + |z|^2)^2}.$$

So the norm of the derivative with respect to the spherical metric is

$$\|f'\|(z) := |f'(z)|(1 + |z|^2)/(1 + |f(z)|^2).$$

In this paper we study the quantity $K(f) = \max_{\overline{\mathbb{C}}} \|f'\|$. d'Ambra and Gromov [1] proposed to study the rate of growth of $\sup \|(f^n)'\|$ as $n \rightarrow \infty$ for the iterates f^n of smooth maps of Riemannian manifolds, especially those maps in a given class for which this growth rate is the smallest possible.

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Such maps are called “slow”. Slow maps of an interval and slow Hamiltonian diffeomorphisms of a 2-torus have been investigated in [3, 16] and [2].

Let f be a rational function of degree d . As the map $f : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ is d -to-1, we conclude that

$$\int \int_{\mathbf{C}} \|f'\|^2 dA = d(f) \int \int_{\mathbf{C}} dA.$$

This implies that

$$K(f) \geq \sqrt{d(f)} \tag{1}$$

We ask how small can $K(f)$ be for a function of given degree.

1. It is known that $K(f) \geq 2$ for all rational functions of degree at least 2. In fact this holds for all smooth maps of the sphere into itself which satisfy $\deg(f) \notin \{0, 1, -1\}$ [10]. It is not known whether $K(f) = 2$ can hold for rational functions of degrees 3 or 4.

2. An interesting question is whether (1) is best possible in certain sense. We have

Theorem 1. *There exists an absolute constant C with the following property. For every $d \geq 2$ there exists a rational function of degree exactly d such that*

$$K(f) \leq C\sqrt{d}.$$

An analogous result was obtained by Gromov [10, Ch. 2D] for smooth maps of spheres of arbitrary dimension.

Littlewood [14] and Hayman [12] studied the quantity

$$\phi(d) = \sup_{\deg f=d} \sup_{R>0} \frac{1}{R} \int \int_{|z|\leq R} \frac{|f'(z)|}{1 + |f(z)|^2} dxdy,$$

where the sup is taken over all rational functions of degree d . For polynomials $\phi(d)$ was also studied in [9, 6, 13]. It is easy to see that $\phi(d) \leq \pi\sqrt{d}$ and Hayman obtained $\phi(d) \geq c_1\sqrt{d}$ using a rational approximation of elliptic functions. Our Theorem 1 implies this with a more elementary proof. Indeed, by a change of the independent variable,

$$\phi(d) = \sup_{\deg f=d} \int \int_{|z|\leq 1} \|f'\| \frac{dxdy}{1 + |z|^2}.$$

Denote

$$f^\# = |f'|/(1 + |f|^2). \quad (2)$$

Let f be the function from Theorem 1. By rotating the sphere of the independent variable we may achieve that

$$\int \int_{|z| \leq 1} (f^\#)^2 dx dy \geq \pi d/2,$$

because the spherical area of the image sphere is π and it is covered d times. Let $M = \max_{|z| \leq 1} f^\#$. Theorem 1 implies that $M \leq C\sqrt{d}$, so

$$\begin{aligned} \phi(d) &\geq \int \int_{|z| \leq 1} f^\# dx dy \geq M^{-1} \int \int_{|z| \leq 1} (f^\#)^2 dx dy \\ &\geq C^{-1} d^{-1/2} \pi d/2 = \pi C \sqrt{d}/2. \end{aligned}$$

Proof of Theorem 1. For every positive integer n , consider the function

$$f_n(z) = \prod_{k=-n}^n \tanh(z + 2k).$$

We first show that

$$\frac{|f_n|}{1 + |f_n|^2} \leq C, \quad (3)$$

where C is independent of n . We have

$$||\tanh z| - 1| \leq 4e^{-2x} < 1, \quad x = |\Re z| \geq 1. \quad (4)$$

Now fix $z \in \mathbf{C}$ and let m be an integer such that $|2m - \Re z| \leq 1$. For $|w - z| \leq 1/2$ put

$$g(w) = \begin{cases} \tanh(w - 2m), & \text{if } |m| \leq n, \\ 1, & \text{otherwise} \end{cases}$$

Then evidently

$$\frac{|g'(w)|}{1 + |g(w)|^2} \leq \frac{|\tanh'(w)|}{1 + |\tanh(w)|^2} \leq C_0. \quad (5)$$

We write $f_n = gh$, $f'_n = g'h + gh'$, and estimate h using (4):

$$|h(w)| \leq \prod_{k=0}^{\infty} \left(1 + 4e^{-2(2k+1/2)}\right)^2 =: C_1.$$

Now h is holomorphic in $|w - z| \leq 1/2$, so by Cauchy's theorem,

$$|h'(z)| \leq 4C_1.$$

Next we estimate $h(z)$ from below using (4) again:

$$|h(z)| \geq \prod_{k=0}^{\infty} \left(1 - 4e^{-2(2k+1)}\right)^2 =: C_2.$$

Combining all these estimates we obtain

$$\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq \frac{C_1|g'(z)| + 4C_1|g(z)|}{1 + C_2^2|g(z)|^2} \leq C_2^{-2}(C_1C_0 + 4C_1).$$

This proves (3)

When $|\Re z| \geq n + 1$, we will obtain better estimates. Using

$$|\tanh'(z)| = |\cosh(z)^2| \leq 16e^{-2|x|}, \quad |x| = |\Re z| \geq 1,$$

we obtain for $z > 2n$ and $\xi = \Re z - 2n$:

$$|f'_n(z - 2n)| \leq C_1 \sum_{k=0}^{\infty} |\cosh(\xi + 2k)|^2 \leq 16C_1 \sum_{k=0}^{\infty} e^{-2(\xi+2k)} = 16C_1C_3e^{-2\xi},$$

which gives

$$\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq Ce^{-2(|x|-n)} \leq Ce^{-2|x|/n}$$

if $|x| \geq n + 2$. Combining this with (3) we obtain

$$\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \leq Ce^{-2|x|/n}$$

for all z . As f_n has period π , there exists a rational function R_n such that $f_n(nz) = R_n(e^{2z})$. This rational function has degree $2n^2$ and the derivative satisfies $\|R'_n\| \leq Cn$. This completes the proof of Theorem 1.

Theorem 2. *There exists an absolute constant $c > 1$ with the property that*

$$K(f) \geq c\sqrt{d}$$

for all rational functions of degree $d \geq 2$.

This can be considered as an analog of a result of Tsukamoto [18]. He studied spherical derivatives (2) of meromorphic functions $F : \Delta \rightarrow \overline{\mathbf{C}}$, where Δ is the unit disc with the Euclidean metric and proved that there exists an absolute constant $c_1 < 1$ with the property that $\omega(F(\Delta)) \leq c_1\pi$ for all meromorphic functions F satisfying $F^\# \leq 1$, where

$$\omega(F(\Delta)) = \int_{|z|<1} (F^\#(z))^2 dx dy.$$

We derive Theorem 2 from this result. In fact we show that Theorem 2 holds with $c = 1/\sqrt{c_1}$.

Proof of Theorem 2. Proving by contradiction, we suppose that there exists a sequence f_m of rational functions of degrees m , such that

$$K(f_m)/\sqrt{m} \rightarrow b < 1/\sqrt{c_1}. \quad (6)$$

Let ω be the spherical area measure, so that

$$\int_{\overline{\mathbf{C}}} d\omega = \pi,$$

and $\omega_m = f_m^* \omega$ the pull back of ω by f_m . Then

$$\int_{\overline{\mathbf{C}}} d\omega_m = \pi m. \quad (7)$$

It is easy to see that we can find discs $D_m = D(a_m, r_m) \subset \overline{\mathbf{C}}$ (with respect to the spherical metric) of radii r_m such that

$$\int_{D_m} d\omega = \pi/(mb^2) \quad \text{and} \quad \int_{D_m} d\omega_m \geq \pi/b^2. \quad (8)$$

To show this, choose r_m so that the first equation is satisfied and then integrate

$$F(a) := \int_{D(a, r_m)} d\omega_m(a)$$

with respect to a . Evidently $r_m \sim 1/b\sqrt{\pi m}$, $m \rightarrow \infty$.

Let a'_m be the point diametrically opposite to a_m , and let $\phi_m : \mathbf{C} \rightarrow \overline{\mathbf{C}} \setminus \{a'_m\}$ be the conformal map (inverse to a stereographic projection) such that $\phi_m(0) = a$, $\phi_m(\Delta) = D_m$, then

$$\phi_m^\#(z) \leq r_m(1 + o(1)), \quad m \rightarrow \infty, \quad (9)$$

uniformly with respect to z . Then $F_m = f_n \circ \phi_m$ is a normal family. Let $F = \lim F_m$. From (6), (9) follows that then $F^\# \leq 1$, but the area of $F(\Delta) \geq \pi/b^2$ in view of (8), contradicting the result of Tsukamoto.

Theorems 1 and 2 have analogs for maps $\mathbf{P}^1 \rightarrow \mathbf{P}^n$ which are stated and proved in the same way as for $n = 1$, using the Fubini–Study metric for the norm of the derivative. Constants C and c will of course depend on n .

Now we consider dynamical questions. By f^n we denote the n -th iterate, and our standing assumption is that $d(f) \geq 2$. We define

$$k_\infty(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log K(f^n).$$

The limit always exists because the sequence $a_n = \log K(f^n)$ is subadditive, $a_{m+n} \leq a_m + a_n$ and for every such positive sequence the limit $\lim_{n \rightarrow \infty} a_n/n$ exists and is equal to $\inf_n a_n/n$ (see, for example, Lemma 1.16 in [4]).

It follows that $k_\infty(f^m) = mk_\infty(f)$.

Notice that k_∞ is independent of the choice of a smooth Riemannian metric on the sphere, and is invariant under conjugation by conformal automorphisms. Obviously, $k_\infty(f) \leq \log K(f)$.

3. What is the smallest value of $k_\infty(f)$ for rational functions of given degree?

The trivial lower estimate of $K(f)$ gives

$$k_\infty(f) \geq (1/2) \log d(f). \quad (10)$$

We will see that equality never happens, and that the Lattés functions are not extremal for minimizing k_∞ . For functions f_d of degree d from Theorem 1 we have

$$k_\infty(f_d)/\log d \rightarrow 1/2, \quad d \rightarrow \infty,$$

so the $(1/2)$ in (10) cannot be replaced with a larger constant.

In [7] these quantities were studied for polynomials, in particular, the inequality $k_\infty(f) \geq \log d(f)$ was established for polynomials, with equality only if f is conjugate to z^d .

Let us consider a slightly different quantity, the *maximum characteristic exponent*

$$\chi_m(f) = \sup_z \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|(f^n)'(z)\| \leq k_\infty(f). \quad (11)$$

The difference in the definitions of k_∞ and χ_m is in the order of \max_z and $\lim_{n \rightarrow \infty}$. Przytycki proved in 1998 (reproduced in [17]) that the same quantity $\chi_m(f)$ can be obtained by taking the sup over periodic points z of f , in which case the lim sup in (11) can be of course replaced by the ordinary limit. Moreover, he proved the following:

Theorem P. *For every $\epsilon > 0$ there exists a periodic point z such that*

$$\frac{1}{m} \log \|(f^m)'(z)\| \geq k_\infty - \epsilon, \quad (12)$$

where m is a period of z .

In particular, $k_\infty = \chi_m$, and one can replace $\sup_{z \in \overline{\mathcal{C}}}$ in (11) by sup over all periodic points.

4. Let $\chi_a(f)$ be the average value of

$$\log \|(f^n)'(z)\|$$

over the measure μ of maximal entropy. According to the multiplicative ergodic theorem, we have

$$\chi_a(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(f^n)'(z)\|$$

almost everywhere with respect to μ .

Theorem 4. $\chi_a(f) \geq (1/2) \log d$, this is best possible and equality holds only for Lattés examples.

Proof. The estimate follows from the formula

$$\chi_a(f) = h(f) / \dim \mu,$$

where $h(f) = \log d$ is the topological entropy, and $\dim \mu$ is the Hausdorff dimension of the maximal measure, see, for example [8] and references therein. Obviously $\dim \mu \leq 2$ so we obtain our inequality. On the other hand, a theorem of A. Zdunik [20] says that $\dim \mu = 2$ can happen only for Lattés examples. This completes the proof of the theorem.

Now we consider the Lattés functions [15].

Proposition 1. *If L is a Lattés function then $\chi_m(L) \geq \log d(L)$.*

Proof. A Lattés function can be defined by a functional equation

$$F(\lambda z) = L \circ F(z), \quad d(L) = \lambda^2,$$

where F is an elliptic function with a critical point at 0. Assuming without loss of generality that $F(0) = 0$ we conclude that 0 is a fixed point of L . Now the derivative at this fixed point is

$$\lambda \lim_{z \rightarrow 0} F'(\lambda z)/F'(z) = \lambda^2 = d(L),$$

from which follows that $\chi_m(L) \geq d(L)$.

To summarize, we have 4 quantities satisfying inequalities

$$\frac{1}{2} \log d \leq \chi_a(f) \leq \chi_m(f) = k_\infty(f) \leq \log K(f). \quad (13)$$

For Latteés functions we have

$$(1/2) \log d = \chi_a(L) < \chi_m(L) = \log d.$$

In the first inequality equality holds only for Lattés functions. For functions f_d constructed in Theorem 1, we have $\log K(f_d) \leq (1/2) \log d + O(1)$.

Which of the rest of these inequalities (13) are strict and what are the conditions of equality? According to a private communication of Przytycki, the method of Zdunik can be used to show that $\chi_a(f) = \chi_m(f)$ is only possible when f is conjugate to $z^{\pm d}$ in which case both quantities are equal to $\log d$.

Proposition. *Let $M = \{z : \|f'(z)\| = K(f)\}$. Then $k_\infty = \log K(f)$ if and only if M contains a cycle.*

Proof. If z is a point whose trajectory is in M , then $\|(f^m)'(z)\| = \|f'(z)\|^m$ so $k_\infty(f) = \log K(f)$.

Suppose now that $k_\infty = \log K(f)$. We claim that

$$\bigcap_{j=0}^{\infty} f^j(M) \neq \emptyset.$$

Indeed, otherwise for some m we have

$$\bigcap_{j=0}^m f^j(M) = \emptyset,$$

and if this holds, then

$$\begin{aligned} mk_\infty(f) &= k_\infty(f^m) \leq \log K(f^m) = \log \max_z \prod_{j=0}^{m-1} \|f'(f^j(z))\| \\ &< m \log K(f), \end{aligned}$$

contrary to our assumption. This proves the claim.

The set M_∞ is forward invariant. All sets

$$\bigcap_{j=0}^m f^j(M)$$

are real algebraic subsets of $\overline{\mathbf{C}}$, and thus the set M_∞ is also real algebraic. If M_∞ is finite it must contain a cycle.

Suppose that M is infinite. Let M_1, \dots, M_q be the list of connected components of M . Then there must be a cycle of components. Let M_1, \dots, M_k be this cycle, so that $f(M_j) \subset M_{j+1}$, $f(M_k) = M_1$. If any of these components contains singular points, then all singular points in these components form a finite invariant set, and this set must contain a cycle. If all M_j , $1 \leq j \leq k$ are smooth, then $f^k : M_1 \rightarrow M_1$ is an expanding map of a circle, so it must have a fixed point. Again we obtain a cycle in M . This completes the proof.

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